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Modeling ENSO-Driven Climate Variability and Extreme Weather: Hopf Bifurcations and Control of Ocean–Atmosphere Oscillations

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Lakshmi. N. Sridhar

Chemical Engineering Department

University of Puerto Rico

Mayaguez, PR 00681

***Corresponding author**: Lakshmi. N. Sridhar.

Abstract

The El Niño–Southern Oscillation (ENSO) is a dominant driver of interannual climate variability, strongly influencing global weather patterns, tropical cyclone activity, and extreme climate events. This study develops and analyzes a reduced-order nonlinear ocean–atmosphere model that captures the essential mechanisms underlying ENSO dynamics, focusing on the interaction between sea surface temperature anomalies and thermocline depth anomalies in the tropical Pacific. The model incorporates positive feedback between ocean and atmosphere, delayed negative feedback through ocean adjustment processes, and nonlinear saturation effects that constrain system growth. These interactions give rise to self-sustained oscillations representing El Niño and La Niña cycles. A key feature of the system is the emergence of a Hopf bifurcation, marking the transition from a stable climate state to periodic climate variability. This transition is characterized through analytical and numerical Jacobian analysis, confirming the onset of oscillatory behavior. The resulting limit cycles provide a simplified representation of ENSO-driven climate oscillations that modulate global weather extremes, including variations in rainfall, drought patterns, and tropical cyclone activity across multiple ocean basins. The framework further highlights how small changes in ocean–atmosphere coupling strength can significantly alter climate stability, emphasizing the sensitivity of the tropical climate system. Overall, the model provides a physically interpretable and mathematically tractable description of ENSO dynamics, offering insight into the mechanisms governing climate variability and its far-reaching impacts on global weather and extreme events.

Keywords: ENSO dynamics, ocean–atmosphere interaction, Hopf bifurcation, climate variability, tropical cyclones

Introduction

The El Niño–Southern Oscillation (ENSO) is the dominant mode of interannual climate variability in the Earth system and arises from coupled interactions between the tropical Pacific Ocean and atmosphere. It is characterized by alternating warm (El Niño) and cold (La Niña) phases, associated with anomalous sea surface temperatures, thermocline displacements, and changes in large-scale atmospheric circulation. These coupled anomalies reorganize tropical convection and modify global atmospheric teleconnections, making ENSO a primary driver of year-to-year variability in weather and climate extremes worldwide. Because of its far-reaching influence, ENSO is not only a fundamental scientific problem in climate dynamics but also a major factor in disaster risk, particularly for droughts, floods, heatwaves, and tropical cyclone activity.

One of the most significant impacts of ENSO is its strong modulation of tropical cyclone activity, especially in the Atlantic basin. During El Niño events, enhanced convection over the central and eastern Pacific strengthens upper-level westerly winds over the Caribbean and tropical Atlantic, increasing vertical wind shear and suppressing hurricane formation. This dynamical environment inhibits cyclone development and reduces the intensity and frequency of Atlantic hurricanes. In contrast, La Niña conditions weaken vertical wind shear and promote warmer Atlantic sea surface temperatures, creating favorable conditions for enhanced hurricane activity. As a result, ENSO phases are strongly correlated with seasonal hurricane variability and are widely used in operational seasonal hurricane forecasts. This relationship highlights the importance of understanding ENSO dynamics not only for climate theory but also for practical forecasting and risk management.

Beyond the Atlantic basin, ENSO significantly influences tropical cyclone behavior across the Pacific and Indian Oceans. El Niño events tend to shift cyclone activity eastward into the central and eastern Pacific, increasing storm frequency in regions that are typically less active, while simultaneously suppressing cyclone formation in the western Pacific. This redistribution is linked to large-scale changes in the Walker circulation and shifts in the location of maximum sea surface temperature anomalies. Consequently, ENSO acts as a global organizer of tropical cyclone climatology, altering storm genesis regions, tracks, and intensities across multiple ocean basins. These teleconnections underscore the interconnected nature of the climate system, where perturbations in one region can have far-reaching impacts on weather extremes worldwide.

ENSO also plays a critical role in shaping global precipitation patterns and hydrological extremes. El Niño events are often associated with increased rainfall and flooding in parts of western South America, while simultaneously inducing drought conditions in Australia, Indonesia, and parts of Southeast Asia. La Niña events typically produce opposite anomalies, reinforcing the role of ENSO as a driver of alternating wet and dry climate extremes. These precipitation anomalies are mediated through atmospheric wave propagation, changes in convection, and shifts in jet stream positioning, all of which are directly linked to tropical Pacific sea surface temperature anomalies. The resulting impacts on agriculture, water resources, and infrastructure make ENSO one of the most economically and socially significant climate phenomena.

From a dynamical systems perspective, ENSO is often interpreted as a self-sustained oscillatory phenomenon arising from nonlinear ocean–atmosphere feedbacks. Key processes include positive feedback between sea surface temperature and atmospheric circulation, delayed negative feedback through thermocline adjustment, and nonlinear saturation mechanisms that limit amplitude growth. These interactions can generate oscillatory behavior even in the absence of external periodic forcing. In simplified conceptual models, these mechanisms are often represented using low-order dynamical systems that capture the essential structure of ENSO variability. Such models are particularly valuable because they allow analytical investigation of stability, oscillation onset, and sensitivity to parameter changes.
In this context, the transition from stable climate equilibrium to oscillatory ENSO behavior can be understood as a qualitative change in system dynamics driven by variations in ocean–atmosphere coupling strength. Near critical

thresholds, small perturbations in system parameters can lead to large-amplitude oscillations, indicating heightened climate sensitivity. This behavior is consistent with the presence of nonlinear instabilities in the coupled system, which can give rise to sustained periodic variability in the form of El Niño and La Niña cycles. Understanding these transitions is essential for improving predictability and assessing the stability of the tropical climate system under natural variability and external forcing.

In recent decades, increasing attention has been given to how climate change may influence ENSO behavior and its associated teleconnections. Observational and modeling studies suggest that changes in background sea surface temperatures, atmospheric stratification, and circulation patterns may alter the frequency, amplitude, and spatial structure of ENSO events. In particular, there is evidence that extreme El Niño events may become more frequent under greenhouse warming scenarios, potentially amplifying global climate impacts. Such changes would have profound implications for hurricane risk, as stronger El Niño conditions are typically associated with suppressed Atlantic hurricane activity but enhanced Pacific cyclone activity. However, significant uncertainties remain regarding the precise nature of these changes and their dependence on model structure and internal variability.

Given the central role of ENSO in modulating extreme weather events and global climate variability, there is strong motivation to develop simplified yet physically meaningful models that can capture its essential dynamics. Reduced-order nonlinear models provide a powerful framework for studying the mechanisms underlying ENSO oscillations, particularly the interplay between instability, coupling, and nonlinear saturation. These models also enable the application of advanced mathematical tools from nonlinear dynamics, such as stability analysis and bifurcation theory, to identify critical transitions in the climate system.

Building on this foundation, the present study develops a low-order ENSO-type model that captures the essential feedbacks between sea surface temperature and thermocline depth. The model is used to investigate the emergence of oscillatory behavior through nonlinear interactions and to explore how external modulation of the ocean–atmosphere coupling influences system stability. In addition, the framework provides a basis for examining how climate variability can be regulated through controlled modifications of system parameters, offering insight into potential pathways for reducing extreme variability in a conceptual climate setting. Through this approach, the study aims to connect fundamental concepts in nonlinear dynamics with pressing questions in climate variability, predictability, and extreme weather impacts, particularly those related to ENSO-driven hurricane variability and global climate teleconnections.

Literature Review

ENSO dynamics and its predictability have been extensively studied as a coupled ocean–atmosphere phenomenon driven by tropical Pacific feedback mechanisms [1]. Early theoretical developments introduced unified oscillator frameworks that formalized ENSO as a self-sustained climate mode arising from coupled feedbacks between ocean and atmosphere [2]. Subsequent work emphasized the role of ocean–atmosphere interaction processes in regulating tropical climate variability and establishing large-scale teleconnections across the Pacific basin [3]. Nonlinear features such as asymmetry between El Niño and La Niña phases were identified, highlighting the importance of nonlinear coupling terms in ENSO evolution [4]. A comprehensive physical understanding of ENSO was further synthesized through classical ocean–atmosphere interaction theory, linking thermocline adjustment and wind stress feedback mechanisms [5]. Linear and nonlinear oscillator models were then extended to explain periodicity and phase locking behavior in ENSO cycles [6].

The introduction of delayed feedback representations provided a new perspective on ENSO variability by capturing memory effects in ocean adjustment processes [7]. Climate change studies later demonstrated that global warming may alter ENSO stability and modify its amplitude and frequency characteristics [1]. Predictability studies emphasized that ENSO forecast skill is fundamentally limited by coupled ocean–atmosphere uncertainties and system sensitivity [9].

Observational and modeling advances confirmed that ENSO remains one of the dominant sources of interannual climate predictability [10]. Climate projections indicated that ENSO variability may intensify under greenhouse warming scenarios, increasing the frequency of extreme events [11]. Extreme El Niño events were identified as potentially becoming more frequent in a warming climate, with significant global climate impacts [12].

Further modeling efforts examined ENSO behavior under changing background climates and highlighted the sensitivity of oscillatory modes to mean state shifts [13]. Additional studies confirmed that ENSO characteristics are strongly influenced by changes in thermocline depth and zonal wind stress distributions [14]. Climate diversity research revealed that ENSO is not a single-mode oscillation but consists of multiple event types with distinct spatial and temporal structures [15]. Updated theoretical frameworks refined classical ENSO paradigms by incorporating nonlinear and stochastic influences [16]. Comprehensive synthesis studies established ENSO as a complex, multi-scale climate phenomenon involving coupled atmosphere–ocean feedbacks and internal variability [17]. Subsequent reviews reinforced the importance of nonlinear interactions in generating ENSO diversity and irregular periodicity [18].

Large-scale climate projections confirmed increasing ENSO variability and its strong sensitivity to anthropogenic forcing [19]. Recharge oscillator theory was revisited and refined to better explain phase transitions in ENSO dynamics [20]. Additional studies demonstrated that ENSO behavior under climate change conditions may exhibit altered stability and amplitude modulation [21]. Stochastic modeling approaches were introduced to better capture observed irregularities in ENSO time series [22]. Comprehensive reviews of ENSO dynamics consolidated understanding of thermocline–SST coupling and atmospheric feedback mechanisms [23]. Further synthesis highlighted the evolving understanding of ENSO in the context of global climate change and its broader Earth system impacts [24].

More recent studies emphasized ENSO extremes and their increasing relevance for climate risk assessment and disaster preparedness [25]. Climate model projections continued to show robust signals of ENSO intensification under greenhouse forcing scenarios [26]. Pantropical studies linked ENSO variability to global climate teleconnections and extreme weather events across multiple continents [27]. Advanced stochastic and reduced-order modeling approaches were developed to rigorously represent ENSO dynamics under uncertainty [28]. Finally, recent research has highlighted the global importance of ENSO teleconnections in regulating precipitation, temperature extremes, and atmospheric circulation patterns worldwide [29].

Main Objectives of this Work

The primary objective of this study is to develop and analyze a simplified ENSO-type ocean–atmosphere model capable of capturing the essential nonlinear dynamics responsible for climate variability in the tropical Pacific. In particular, the work aims to represent the interaction between sea surface temperature anomalies and thermocline depth anomalies using a low-dimensional dynamical system that retains the key physical feedback mechanisms of the real climate system. A central objective is to investigate the emergence of oscillatory climate behavior through nonlinear stability analysis, with emphasis on identifying Hopf bifurcation points that mark the transition from steady-state climate conditions to self-sustained ENSO-like oscillations. This includes characterizing the stability properties of equilibrium solutions using analytical and numerical Jacobian analysis and verifying the onset of complex eigenvalues associated with oscillatory dynamics. Another major objective is to study the structure of the resulting limit cycles beyond the bifurcation point and interpret them in terms of recurring warm and cold climate phases. This helps to establish a mathematical connection between nonlinear dynamical systems theory and observed interannual climate variability. The study further aims to incorporate external forcing via a control parameter that modulates the effective strength of ocean–atmosphere coupling. This allows the system to be analyzed under controlled conditions, enabling the investigation of how external interventions influence stability and oscillatory behavior. A key objective is to formulate and solve an optimal control problem that minimizes climate variability by reducing the amplitude of temperature and thermocline fluctuations while also limiting control effort. This provides a systematic framework for suppressing `undesired oscillations in the climate system. Finally, the work aims to introduce a bifurcation-aware control strategy that incorporates stability constraints into the optimization process, ensuring that the controlled system avoids critical regimes associated with Hopf bifurcations. Through this, the study seeks to demonstrate how nonlinear dynamics, bifurcation theory, and optimal control can be integrated to better understand and potentially regulate climate oscillations such as ENSO. The rest of this paper is organized as follows. First, the model equations are presented, followed by a description of the numerical procedures used. The results discussion and conclusions are then presented. This study presents several novel contributions at the intersection of climate dynamics, nonlinear systems theory, and optimal control. It develops a reduced-order ENSO-type model that explicitly retains the essential mechanisms responsible for Hopf bifurcation and limit cycle oscillations in ocean–atmosphere interactions, providing a mathematically transparent framework for studying stability transitions in the climate system. Unlike purely descriptive approaches, the formulation enables direct analytical and numerical characterization of oscillatory climate behavior through bifurcation theory. A key novelty is the integration of Hopf bifurcation analysis with optimal control, allowing the climate system to be actively steered relative to its intrinsic instability boundary rather than being studied only in a passive sense. The work combines MATCONT-based continuation with analytical Jacobian derivation to rigorously identify and verify the onset of oscillations, ensuring consistency between numerical and theoretical results. Furthermore, it introduces a bifurcation-aware optimal control formulation in which stability constraints are embedded directly into the optimization problem, enabling control strategies that explicitly avoid critical transitions. The incorporation of a neural-network-based surrogate model for stability estimation provides a computationally efficient approximation of eigenvalue-based constraints within the optimization loop, significantly reducing computational cost while preserving dynamical accuracy. The framework also strengthens physical interpretability by directly linking key climate variables such as sea surface temperature and thermocline depth to dynamical systems concepts such as eigenvalue crossings and limit cycles. In addition, the comparative analysis between controlled and uncontrolled regimes quantitatively demonstrates the effectiveness of stability-aware forcing in reducing variability and suppressing oscillatory behavior. The study further bridges climate science and control theory by treating ENSO not only as a natural oscillatory phenomenon but also as a controllable nonlinear dynamical system. The methodology is computationally efficient, fully compatible with modern tools such as Pyomo and IPOPT, and scalable to higher-dimensional climate models. Overall, the work provides a unified framework for understanding and, potentially, controlling climate oscillations through mathematically grounded, bifurcation-aware strategies. The rest of this paper is organized as follows. First, the model equations are presented, followed by a description of the numerical procedures used. The results discussion and conclusions are then presented.

Model Equations

The model represents a low-order nonlinear dynamical system for ENSO variability in the tropical Pacific, capturing the essential interaction between sea surface temperature anomalies and thermocline depth anomalies. It is based on a coupled ocean–atmosphere feedback mechanism in which SST anomalies are amplified by atmospheric responses and modulated by subsurface ocean dynamics. The thermocline variable acts as delayed negative feedback, representing oceanic adjustment processes that redistribute heat vertically and regulate surface temperature evolution. A nonlinear cubic term is included in the SST equation to represent saturation effects arising from physical constraints such as radiative damping, convection limits, and nonlinear mixing. The coupling between the two variables forms a recharge–discharge oscillator structure commonly used in simplified ENSO theory. This interaction allows the system to exhibit self-sustained oscillations under appropriate parameter regimes. The model is capable of reproducing Hopf bifurcation behavior, marking the transition from a stable climatic equilibrium to periodic climate variability. The oscillatory regime corresponds physically to alternating warm and cold phases of ENSO-like behavior. The parameters represent effective large-scale climate feedback strengths rather than localized physical processes, making the system a conceptual rather than predictive model. Despite its simplicity, the model retains the fundamental nonlinear mechanisms responsible for climate variability. It therefore serves as a minimal framework for studying stability, bifurcation, and control of climate oscillations. We begin by constructing a low-order ENSO-type ocean–atmosphere model that captures the essential feedback between sea surface temperature anomalies and thermocline depth anomalies. Let $T(t)$ denote the equatorial Pacific sea surface temperature (SST) anomaly and $h(t)$ denote the thermocline depth anomaly. The evolution of these two coupled variables is derived from simplified physical arguments involving air–sea interaction, ocean adjustment, and nonlinear saturation.

The rate of change of SST anomaly is assumed to be governed by three main processes. First, there is positive feedback between the SST anomaly and the atmospheric response, representing the fact that warmer SST enhances convection and wind stress anomalies that further amplify the temperature field. This feedback is modeled as a linear growth term proportional to T, written as aT, where a is the effective coupling strength between the ocean and atmosphere.
Second, the thermocline depth anomaly influences the SST through vertical heat exchange processes. A deeper thermocline generally suppresses upwelling of cold water, thereby increasing SST, while a shallower thermocline enhances cooling. This restoring interaction is represented as a linear coupling term $-bh$, where $b>0$ measures the strength of thermocline feedback on surface temperature.
Third, nonlinear saturation effects limit unbounded growth of SST anomalies. Physically, this reflects the fact that convection, radiative damping, and nonlinear ocean mixing act to stabilize extreme temperature departures. This effect is modeled using a cubic damping term $-cT^3$, where $c>0$ ensures bounded dynamics and prevents divergence of solutions.
Combining these contributions, the SST evolution equation becomes

$$\frac{dT}{dt} = aT - bh - cT^3. \quad (1)$$

The thermocline dynamics are derived from large-scale ocean adjustment processes. Variations in SST drive wind stress anomalies that induce changes in ocean wave propagation and thermocline displacement. This leads to a recharge–discharge mechanism in which warm SST anomalies correspond to a deepening thermocline and vice versa. This coupling is represented by a linear forcing term dT, where $d>0$ measures the efficiency of SST-induced thermocline adjustment.
Simultaneously, the thermocline relaxes toward its mean climatological state through dissipative processes such as friction, wave radiation, and mixing. This relaxation is modeled as a linear damping term $-eh$, where $e>0$ represents the relaxation rate.
Thus, the thermocline evolution equation is given by

$$\frac{dh}{dt} = dT - eh \quad (2)$$

The coupling parameter a is not treated as a fixed constant but is instead decomposed into a baseline atmospheric–ocean coupling strength and an externally imposed control input. This reflects the idea that large-scale forcing mechanisms, such as changes in atmospheric circulation intensity or external interventions, can modulate the effective feedback strength. Hence, we write

$$a = a_0 + u(t), \quad (3)$$

where a_0 is the natural coupling parameter and $u(t)$ is a time-dependent control input.
Substituting this expression into the governing equations yields the controlled ENSO system

$$\frac{dT}{dt} = \left(a_0 + u(t)\right)T - bh - cT^3$$
$$\frac{dh}{dt} = dT - eh \qquad (4)$$

This system represents a minimal nonlinear oscillator in which oscillatory behavior arises from the interaction between positive feedback (instability), cross-variable coupling (ocean–atmosphere interaction), and nonlinear saturation (stabilization). The competition between these mechanisms produces Hopf bifurcations and limit cycle oscillations, corresponding physically to the transition between stable climate conditions and self-sustained ENSO variability. Tables 1 and 2 give the variable and parameter details

Table 1: Model Variables

Symbol	Variable Name	Physical Meaning	Units
T	Sea Surface Temperature anomaly	Deviation of SST from climatological mean (ENSO warming/cooling)	°C
h	Thermocline depth anomaly	Deviation of thermocline depth from mean position	m
$u(t)$	Control input	External modulation of ocean–atmosphere coupling strength	dimensionless
t	Time	Temporal evolution variable	days / months (nondimensional in model)

Table 2: Model Parameters

Symbol	Parameter Name	Value	Physical Meaning	Units
a_0	Baseline coupling strength	1.8	Natural ocean–atmosphere feedback intensity	1/time
b	Thermocline feedback coefficient	1.0	Strength of thermocline influence on SST	1/time
c	Nonlinear saturation coefficient	1.0	Limits growth of SST anomalies (damping nonlinearity)	1/(°C²·time)
d	Recharge coefficient	1.9	SST forcing effect on thermocline adjustment	1/time
e	Thermocline damping rate	1.0	Relaxation of thermocline to equilibrium	1/time

Bifurcation Analysis and Optimal Control

Bifurcation Analysis

Continuation and bifurcation computations were carried out using the MATLAB-based software MATCONT. Bifurcation analysis provides important insights into the mechanisms underlying the existence of multiple steady states and self-sustained oscillations in nonlinear dynamical systems. In particular, branch points and limit points are associated with the emergence of multiple steady-state solutions, whereas oscillatory dynamics and periodic solutions originate from Hopf bifurcations. The package, originally developed and subsequently enhanced by several researchers, enables the systematic identification of limit points (LP), branch points (BP), and Hopf bifurcation points (H) in nonlinear dynamical models [30]. This program identifies Limit points (LP), branch points (BP), and Hopf bifurcation points(H) for a system of ordinary differential equations

$$\frac{dx}{dt} = f(x, \alpha) \quad (5)$$

$x \in R^n$ where the bifurcation parameter is α .

At a limit point, the $n+1$ th component of the tangent vector $w_{n+1} = 0$. For a branch point,

the matrix $B = \begin{bmatrix} A \\ w^T \end{bmatrix}$ must be singular and have a determinant of 0.

At a Hopf bifurcation point,

$$\det(2 f_x(x, \alpha) @ I_n) = 0 \quad (6)$$

@ indicates the bialternate product while I_n is the n-square identity matrix. More details can be found in Kuznetsov and Govaerts respectively [31,32].

Optimal Control

Pyomo.dae[33] is used for the Optimal Control calculations. Pyomo with its Pyomo.DAE module provides an efficient framework for the formulation and solution of dynamic optimization problems involving systems of differential and algebraic equations. The package provides a symbolic modeling environment for representing differential-algebraic equation (DAE) systems in optimization applications. In Pyomo.DAE, the continuous-time dynamic model is transformed into a nonlinear programming (NLP) problem through discretization techniques such as orthogonal collocation, allowing the resulting optimization problem to be solved using standard NLP solvers. The NLP is solved using IPOPT [34].

Formation of stability Dataset from MATCONT results

A stability dataset was developed from numerical continuation calculations performed in MATCONT. The stability dataset consists of rows, each representing a continuation point from an equilibrium branch. Each row contains the state variables, the bifurcation parameter, and a stability measure. The stability measure is a numerical value derived from the Jacobian matrix. The Jacobian matrix is computed numerically at each equilibrium point. The eigenvalues are then computed automatically using MATLAB. The maximum value of the real part of these eigenvalues is then computed as a scalar stability measure.

The stability measure is computed using “eig_real_max = max(real(eigvals));” in MATLAB. The stability measure is a quantitative metric in which negative values indicate locally asymptotically stable equilibria, positive values indicate instability, and a zero crossing indicates a Hopf bifurcation. The stability dataset is then saved as a CSV file. The dataset can then be used in subsequent computational calculations to perform classification or regression to identify stability boundaries or approximate bifurcations.

Neural Network Surrogate for Stability Prediction

Direct embedding of eigenvalue calculations into IPOPT-based optimal control is impractical for several reasons: (i) computing eigenvalues at each time step is computationally expensive, (ii) the mapping from states to the maximum eigenvalue is non-smooth near eigenvalue crossings, and (iii) symbolic differentiation of eigenvalues is challenging.

Prior to neural-network training, all input variables were standardized to improve numerical conditioning and training stability. Let x_{raw}denote the vector of state variables and bifurcation parameters= obtained from the stability dataset. For each input variable j, the training mean $\mu_j$and training standard deviation $\sigma_j$were computed over all training samples. = The training mean for the input variable j is defined as the arithmetic average over all training samples as $\mu_j = \frac{1}{N}\sum_{k=1}^{N} x_j^{(k)}$ and the training standard deviation for the input variable $j$is defined as: $\sigma_j = \frac{1}{N}(\sum_{k=1}^{N} x_j^{(k)} - \mu_j)^2$

The standardized inputs were defined as

$$x_j = \frac{x_{raw,j} - \mu_j}{\max(\sigma_j, \varepsilon_{scale})} \quad (7)$$

where $\varepsilon_{\mathrm{scale}}$is a small positive regularization parameter introduced to prevent numerical singularities associated with extremely small variances. This transformation ensures that all inputs remain properly scaled while avoiding excessively large neural-network activation arguments during optimization. The normalization procedure improves neural-network conditioning and enhances the robustness of gradient-based optimization. The vectors $\mu$and σ computed during training were stored and embedded identically within the Pyomo optimal-control formulation to ensure consistency between neural-network training and deployment.

To overcome these limitations, a feedforward neural network is trained to approximate the maximum eigenvalue as a smooth function of the system state and bifurcation parameter. A typical architecture employs the hyperbolic tangent (tanh) as a smooth activation function. If the input vector is denoted by x, which represents the scaled variables, then the network is defined as:

The vectors μ, σ computed during training were stored and embedded identically within the Pyomo optimal control formulation to ensure consistency between neural network training and deployment.

To avoid repeated eigenvalue computations during optimization, a feedforward neural network is trained to approximate the maximum real eigenvalue as a smooth function of the system states and bifurcation parameter. Using hyperbolic tangent activation functions ensures smooth differentiability required by IPOPT. If the input vector is denoted by x, which are the scaled variables, the network architecture is defined as

$$\begin{aligned} z1 &= tanh\left(W1\,x + b1\right) \\ z2 &= tanh\left(W2\,z1 + b2\right) \\ \lambda\_max\_NN &= W3\,z2 + b3 \end{aligned} \quad (8)$$

where $W_i$and $b_i$denote the weight matrices and bias vectors, respectively. The hidden-layer variables $z_1$and $z_2$represent nonlinear transformations of the input variables and intermediate features. The final output $\hat{\lambda}_{\mathrm{max}}$provides a smooth approximation of the spectral abscissa. Because tanh is infinitely differentiable, the network is fully smooth, guaranteeing the availability of first and second derivatives required by IPOPT. Without biases, the network output would be constrained to pass through the origin, limiting flexibility.

The hidden-layer outputs z1, z2, represent nonlinear combinations of the inputs and previous-layer features, respectively. Each element of $z_1$is a smoothed combination of the original inputs, while each element of z2 encodes more abstract patterns extracted from z1. The final output λ_max_NN provides a smooth approximation of the maximum real eigenvalue, enabling efficient and differentiable stability evaluation within the optimal control problem. The integration into optimal control is done using a soft penalty formulation where we use a smooth_max function that converts λ into a smooth, nonnegative penalty that only "activates" when the system is unstable:

$$s_{\max}(\lambda+\varepsilon_1) = smooth - \max(\lambda+\varepsilon_1) = \frac{(\lambda+\varepsilon_1)+\sqrt{(\lambda+\varepsilon_1)^2+\varepsilon_1}}{2} \quad (9)$$

λ is the neural network's predicted maximum eigenvalue at the current state and parameter, while ε is a small positive safety margin to ensure differentiability. The soft penalty formulation involves the new objective function, where the original objective function $\sum optv(t)$ is modified to $\sum (optv(t)+\alpha_{hopf}.s_{\max}(\lambda+\varepsilon_1))^2$ α_{Hopf} controls how aggressively instability is penalized, and prevents numerical issues at exactly λ = 0 and slightly shifts the stability boundary. When α_{Hopf} is 0, no Hopf constraint is implemented. The goal is to minimize the objective function value while ensuring differentiability for IPOPT and avoiding non-smoothness in the optimization. This approach avoids repeated eigenvalue computations and provides a smooth, differentiable surrogate suitable for gradient-based optimization. Furthermore, this enables the incorporation of stability constraints into optimal control without explicitly computing eigenvalues during optimization.

To facilitate integration into optimal control, a stability dataset is constructed from MATCONT continuation results. At each equilibrium point, the Jacobian is evaluated and its eigenvalues computed. The stability metric is defined as the maximum real part of the eigenvalues ($\lambda_{\max} = \max \Re(\lambda(J))$). Negative values indicate stability, positive values indicate instability, and zero crossings correspond to Hopf bifurcations.

To avoid repeated eigenvalue computations within optimization, a feedforward neural network is trained to approximate the spectral abscissa as a smooth function of the system state and bifurcation parameter. Using hyperbolic tangent activation functions ensures smooth differentiability required by gradient-based solvers. The resulting surrogate provides a differentiable stability indicator suitable for embedding within optimal control formulations.

RESULTS

The bifurcation performed with MATCONT revealed the existence of a Hopf bifurcation point at (T,h,u) values (0.670820 1.274558 0.550000) with a first Lyapunov coefficient = 1.034482e+00. Fig. 1a shows the bifurcation diagram and fid 1b the limit cycle. The analytical Jacobian matrix for the ODE is

$$Ja = \begin{bmatrix} a_0+u-3c(T^*)^2 & -b \\ d & -e \end{bmatrix}. \quad (10)$$

At the Hopf bifurcation point the numerical value of the Jacobian is

$$Jn = \begin{bmatrix} 1 & -1 \\ 1.9 & -1 \end{bmatrix} \quad (11)$$

The eigenvalues of this matrix are 0.9486i and -0.9486i. The presence of the 2 imaginary eigenvalues of opposite sign indicates a Hopf Bifurcation at this point.

The optimal control minimizes the function $\sum optv(t) = \sum (T(t))^2 + h(t))^2 + (u(t))^2$ without and with the Hopf bifurcation constraint. This objective function minimizes the overall kinetic energy of the shear modes, the strength of stratification fluctuations, and the external forcing effort, thereby promoting low-energy, weakly forced, and dynamically stable flow states. To investigate the effect of bifurcation-aware operation, an optimal control problem was solved both with and without a Hopf-bifurcation-avoidance constraint using PYOMO.DAE coupled with IPOPT. PYOMO.DAE with IPOPT was used. When no Hopf constraint was implemented, α_{Hopf} =0, in $\sum (optv(t)+\alpha_{hopf}.s_{\max}(\lambda+\varepsilon_1))^2$ the obtained value of $\sum optv(t)$ was 194.095. For a Hopf constraint, $\alpha_{Hopf}=10$, the obtained value of $\sum optv(t)$ was

84.0059. This shows a considerable decrease in the minimized function, indicating the effect of the Hopf bifurcation constraint. This corresponds to a 56.7 % reduction in the minimized objective function value. Figs 2a, 2b, show the optimal control profiles without Hopf constraint, and Figs 3a, 3b show the optimal control profiles with the Hopf constraint.

Discussion

The results demonstrate that the simplified ENSO ocean–atmosphere model possesses a rich nonlinear dynamical structure characterized by the emergence of self-sustained climate oscillations through a Hopf bifurcation. From a climate dynamics perspective, this finding is particularly significant because many large-scale modes of climate variability arise through the interaction of positive and negative feedback mechanisms that naturally generate oscillatory behavior. The El Niño–Southern Oscillation is one of the most prominent examples of such variability and exerts a major influence on weather and climate patterns across the globe.

The bifurcation analysis performed using MATCONT identified a Hopf bifurcation point at ((T,h,u) = (0.670820,1.274558,0.550000)). At this point the Jacobian matrix possesses a pair of purely imaginary eigenvalues (0.9486i, -0.9486i), indicating the transition from a steady climatic state to a periodically oscillating state. Physically, this transition represents the onset of sustained interactions between sea-surface temperature anomalies and thermocline depth anomalies. In the absence of sufficient damping, these interactions can amplify small perturbations and generate persistent climate oscillations resembling recurrent El Niño and La Niña events.
The appearance of a stable limit cycle beyond the Hopf bifurcation indicates that the system evolves toward a periodic attractor rather than returning to equilibrium. In climate terms, this corresponds to the emergence of recurrent warm and cold phases in the tropical Pacific. Such oscillatory behavior is a defining characteristic of ENSO and is responsible for substantial interannual variability in precipitation, temperature, drought occurrence, and storm activity around the world. Consequently, the bifurcation identified in the present model can be interpreted as a critical threshold separating relatively stable climate conditions from a regime characterized by persistent climate oscillations.

The computed first Lyapunov coefficient of (1.034482) confirms the strongly nonlinear nature of the bifurcation. The existence of nonlinear saturation through the cubic term prevents unbounded growth of temperature anomalies and leads to bounded oscillatory solutions. Similar nonlinear saturation mechanisms are present in real climate systems, where oceanic and atmospheric feedbacks cannot grow indefinitely because of energy constraints and competing physical processes. The resulting periodic solutions therefore provide a physically meaningful representation of large-scale climate variability.

An important aspect of the present work is the incorporation of optimal control into the bifurcation framework. Climate systems often operate near critical thresholds, and external forcing can shift the system toward or away from these thresholds. In the present model the control variable modifies the effective ocean–atmosphere coupling strength, thereby influencing the proximity of the system to the Hopf bifurcation boundary. This interpretation is relevant because anthropogenic forcing, greenhouse-gas-induced warming, changes in atmospheric circulation, and large-scale geoengineering interventions can all alter coupling mechanisms within the climate system.

The optimal control calculations reveal a substantial difference between unconstrained and bifurcation-aware operation. Without the Hopf constraint, the minimized objective function value was 194.095. When the Hopf constraint was enforced, the objective function decreased to 84.0059, corresponding to a reduction of approximately 56.7%. This reduction demonstrates that maintaining the system in a dynamically stable region can significantly improve overall system performance. From a climate perspective, this suggests that operating away from critical bifurcation thresholds may reduce the intensity of oscillatory climate variability while simultaneously lowering the magnitude of external interventions required to regulate the system.

The neural-network-based Hopf constraint incorporated into the optimization procedure plays an important role in achieving this result. Rather than repeatedly computing eigenvalues during optimization, the neural network provides a computationally efficient approximation of the stability boundary obtained from the bifurcation analysis. This allows the optimizer to avoid parameter regions associated with instability while maintaining computational tractability. Such data-driven stability constraints may become increasingly valuable in future climate applications where high-fidelity Earth-system models are too expensive to evaluate repeatedly within large-scale optimization frameworks.

From a weather perspective, the suppression of Hopf-induced oscillations is associated with a reduction in the amplitude of sea-surface temperature and thermocline fluctuations. Since ENSO strongly influences atmospheric circulation, smaller oscillations imply reduced variability in rainfall, drought frequency, flooding events, and storm tracks. El Niño events are often associated with enhanced rainfall in some regions and severe droughts in others, while La Niña events produce nearly opposite effects. Therefore, reducing the amplitude of ENSO-like oscillations can be interpreted as a mechanism for reducing extreme weather variability and enhancing climate predictability.

Another important implication concerns climate resilience. Systems operating close to bifurcation points are highly sensitive to perturbations, meaning that relatively small disturbances can produce large responses. Such sensitivity is undesirable because it increases uncertainty in long-term climate projections and complicates adaptation planning. The Hopf-constrained optimization effectively steers the system away from these sensitive regions, thereby increasing robustness against disturbances. This behavior is analogous to maintaining engineering systems within safe operating limits to avoid instability and performance degradation.

The present study also demonstrates the potential of combining nonlinear dynamics, bifurcation theory, machine learning, and optimal control within a unified climate modeling framework. Traditionally, climate prediction and climate control have been treated as separate research areas. Here, the bifurcation structure of the climate system directly informs the optimization process, enabling stability considerations to be incorporated into decision-making. Such an approach could be extended to more realistic ocean–atmosphere models, delayed ENSO models, coupled climate-carbon cycle systems, or regional climate models where tipping points and critical transitions play a central role.

Overall, the results indicate that Hopf bifurcation analysis provides valuable insight into the mechanisms governing climate oscillations, while optimal control offers a systematic approach for regulating these oscillations. The substantial reduction in the objective function achieved through bifurcation-aware control suggests that explicitly accounting for climate stability boundaries can lead to more efficient and robust management strategies. These findings highlight the importance of integrating nonlinear stability analysis into future climate and weather prediction frameworks, particularly as global climate systems are increasingly subjected to external forcing and environmental change.

This work sits at the intersection of nonlinear dynamics, ocean–atmosphere coupling, and control theory, and its implications extend beyond a reduced-order ENSO model to broader questions in climate variability, predictability, and potential intervention strategies. By identifying and controlling a Hopf bifurcation in a simplified ENSO-type system, the study provides a conceptual framework for understanding how climate oscillations emerge, persist, and can potentially be regulated.

One of the most important impacts is on understanding climate tipping thresholds. Many components of the Earth system—such as monsoon transitions, Atlantic circulation modes, and polar feedback mechanisms—exhibit behavior analogous to bifurcations in dynamical systems. The identification of a Hopf bifurcation in the ENSO model highlights that climate variability is not purely stochastic, but often structured by underlying nonlinear instabilities. In this context, the Hopf point represents a critical threshold separating a quiescent climate regime from one characterized by sustained oscillations. Recognizing such thresholds in simplified models provides insight into where real climate systems may exhibit heightened sensitivity to perturbations.

A second major impact lies in improving climate predictability. Near a Hopf bifurcation, the system becomes highly sensitive to parameter variations, which corresponds physically to increased uncertainty in forecasts of El Niño and La Niña events. By explicitly mapping the stability boundary and quantifying how control modifies proximity to the bifurcation, this framework offers a structured way to assess predictability regimes. In particular, regimes close to the bifurcation are associated with amplified variability and reduced forecast skill, while controlled regimes farther from the bifurcation exhibit more stable and predictable dynamics.

The introduction of optimal control into a climate oscillation model represents another significant contribution. Instead of treating ENSO variability as an uncontrollable natural phenomenon, this framework demonstrates that it can be influenced through modulation of key coupling parameters. Although the control variable in this study is abstract, it can be interpreted physically as representing large-scale changes in air–sea coupling strength, heat flux exchange, or effective feedback intensity. The reduction in the objective function under Hopf-constrained optimization indicates that steering the system away from instability reduces both state variability and energetic cost, suggesting a pathway toward stabilizing undesirable climate oscillations in theoretical settings.

This work also has implications for the emerging field of climate intervention and geoengineering assessment. While the present model is highly idealized, it provides a mathematical analogy for how external forcing could shift the climate system relative to critical thresholds. In particular, it shows that inappropriate forcing may push the system toward bifurcation points, increasing the likelihood of large-amplitude oscillations, whereas carefully designed interventions could maintain the system in a dynamically stable regime. This highlights the importance of stability-aware design in any future climate intervention strategy.

Another key impact is methodological. The integration of bifurcation analysis, numerical continuation, and optimal control within a unified framework provides a powerful template for studying complex climate systems. The use of MATCONT to identify bifurcation structures, combined with Pyomo-based optimization constrained by stability information, demonstrates how dynamical systems theory can be embedded directly into control-oriented climate modeling. This approach can be extended to more complex Earth system components, including coupled ocean–atmosphere general circulation models, reduced climate emulators, and data-driven hybrid models.

Finally, the incorporation of a neural-network-based stability constraint introduces a pathway toward data-driven climate stability control. This is particularly relevant in modern climate science, where high-fidelity simulations are computationally expensive and direct stability analysis is often impractical. By learning the relationship between state variables, control inputs, and stability indicators, surrogate models can enable rapid evaluation of whether a system is approaching a tipping point. This opens the door to real-time stability-aware decision frameworks for climate risk assessment.

Overall, the impact of this research lies in demonstrating that climate oscillations such as ENSO can be interpreted through the lens of nonlinear bifurcation theory and, importantly, that their behavior can be systematically influenced through optimal control strategies. While the present model is conceptual, the methodology provides a foundation for future work to understand and manage variability in more realistic climate systems, particularly in regimes where tipping points and nonlinear feedbacks dominate system behavior.

CONCLUSIONS

This study developed and analyzed a reduced-order ENSO-type ocean–atmosphere model to investigate the onset of climate oscillations through nonlinear dynamics and to explore the possibility of regulating such oscillations using optimal control. The governing system, consisting of coupled equations for sea surface temperature anomaly (T) and thermocline depth anomaly (h), exhibits a classical nonlinear feedback structure with cubic saturation and linear coupling. Through bifurcation analysis using MATCONT, the system was shown to undergo a Hopf bifurcation at a critical parameter value, where the Jacobian matrix possesses a pair of purely imaginary eigenvalues. This transition marks the onset of self-sustained oscillations corresponding to recurrent El Niño–La Niña-like behavior.

Beyond the bifurcation point, the system evolves toward a stable limit cycle, confirming the emergence of sustained periodic climate variability. The analytical and numerical Jacobian analysis verified the stability transition, while eigenvalue computations confirmed the Hopf condition. These results demonstrate that even low-order deterministic models can reproduce essential features of climate oscillations observed in the tropical Pacific, highlighting the fundamental role of nonlinear feedback mechanisms in governing ENSO dynamics.

The study further incorporated an optimal control framework in which the atmospheric–ocean coupling parameter is modulated through a control input. The objective function was designed to minimize the combined magnitude of state variables and control effort, thereby reducing the overall energy of climate variability. A bifurcation-aware formulation was introduced by incorporating a Hopf stability constraint derived from the spectral properties of the system. The results show a substantial reduction in the objective function when the Hopf constraint is enforced, indicating that maintaining the system away from critical bifurcation thresholds leads to more stable and energetically efficient dynamics.

Overall, the findings highlight the importance of nonlinear stability analysis in understanding climate variability and demonstrate the potential of optimal control strategies in regulating oscillatory behavior in coupled climate systems. The framework presented here provides a foundation for future extensions to more complex climate models, including delayed feedback systems, spatially extended dynamics, and higher-fidelity Earth system representations.

Declarations

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Authors Contribution
There is only one author who did everything

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All data used is presented in the paper

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